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A NOTE ON SIGN BALANCED INDEX SET OF A GRAPH

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Abstract: Let G be a graph with vertex set V and edge set E. Let g be a labeling from E to $\{+, -\}$. The edge labeling g induces a vertex labeling $h : V \to \{+, -\}$ defined by $h(v) = \prod g(uv)$ for u in $N(v)$, where $N(v)$ is the set of vertices adjacent to v. Let $e(+) = card\{e \in E : g(e) = +\}$, $e(-) = card\{e \in E : g(e) = -\}$ and $v(+) = card\{v \in V : h(v) = +\}$, $v(-) = card\{v \in V : h(v) = -\}$. A labeling g is said to be sign friendly if $\mid e(+) - e(-) \mid \leq 1$. The sign balanced index set $(SBIS)$ of a graph G is defined by $\{\mid v(+) - v(-) \mid :$ the edge labeling g is sign friendly$\}$. In this paper we completely determine the sign balanced index sets of some important family of graphs.

Keywords and Phrases: Edge labeling, sign-friendly, sign balance index set.

2020 Mathematics Subject Classification: 05C78.

1. Introduction

A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions. Graph labelings were first introduced in the mid 1960's. Graph labeling was used in many applications like coding theory, x-ray crystallography, circuit designs and communication network etc. For more details one may refer [7], [3], [5], [6], [8], [2].

Recently Mukti Acharya [1] introduced a new labeling called C-Cordial labeling in signed graphs and she studied C-Cordial labeling for path, cycle and star graph. Now in this paper we extend the same concept to more generalized set called sign balanced index set of some important family of graphs.

Let G be a graph with vertex set V and edge set E. Let g be a labeling from E to $\{+, -\}$. The edge labeling g induces a vertex labeling $h : V \to \{+, -\}$ defined by $h(v) = \prod g(uv)$ for u in $N(v)$, where $N(v)$ is the set of vertices adjacent to v. Let $e(+) = card\{e \in E : g(e) = +\}$ and $e(-) = card\{e \in E : g(e) = -\}$. $v(+) = card\{v \in V : h(v) = +\}$ and $v(-) = card\{v \in V : h(v) = -\}$. A labeling g is said to be sign friendly if $\mid e(+) - e(-) \mid \leq 1$. The sign balanced index set $(SBIS)$ of a graph G is defined by $\{\mid v(+) - v(-) \mid :$ the edge labeling g is sign friendly$\}$.

For convenience, a vertex(edge) is called positive vertex if its label is '+' and negative vertex (edge), if its label is '−'. In this paper we completely determine the sign balanced index sets of $C_n \times P_3$, double triangular snake graph and $\mu(P_n)$.

Before determining the sign balanced index sets, we prove some properties regarding the index numbers and $v(-)$.

Theorem 1.1. *If the number of vertices in a graph G is even then the sign balance index set contains only even numbers.*

Proof. Let $G(V, E)$ be a graph with $|V| = n$, which is even. Now suppose that $v(-) \geq v(+)$ then by the definition of sign balanced index set we have

$$v(-) + v(+) = n$$
$$v(-) - v(+) = r$$

where r be the element of sign balanced index set. Now adding above equations we get

$$2v(-) = n + r.$$

Theorem 1.2. *If the number of vertices in a graph G is odd then the sign balanced index set contains only odd numbers.*

Theorem 1.3. *In sign balanced labeling, $v(-)$ is always even.*

Proof. Let $b_i = |\{v \in V :$ number of negative edges incident on v is equal to $i\}|$, $i = 1, 2, 3, \ldots, n$. Then we have $b_1 + 2b_2 + 3b_3 + \cdots + nb_n$ is equal to twice the number of negative edges, which is even. Therefore $b_1 + 3b_3 + \cdots + nb_n$ is even if n is odd and $b_1 + 3b_3 + \cdots + (n-1)b_{n-1}$ is even if n is even. Which implies $b_1 + b_3 + \cdots + b_n = v(-)$ is even if n is odd and $b_1 + b_3 + \cdots + b_{n-1} = v(-)$ is even if n is even.

2. Sign Balanced Index Set of $C_n \times P_3$

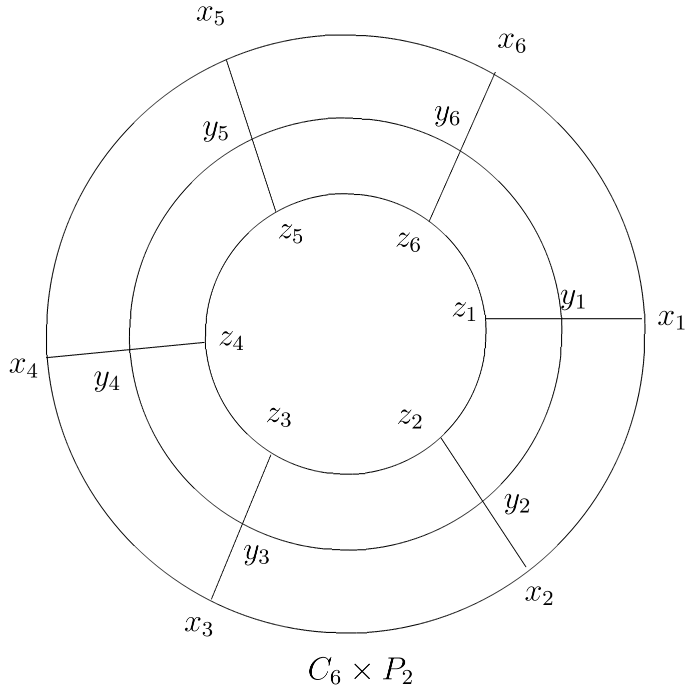

Figure 1

We consider the problem of finding $SBIS(C_n \times P_3)$ into four cases namely,

$$\begin{array}{ll} n \equiv 0(mod\ 4) & n \equiv 1(mod\ 4) \\ n \equiv 2(mod\ 4) & n \equiv 3(mod\ 4) \end{array}$$

Theorem 2.1. *If $n \equiv 0(mod\ 4)$, then $SBIS(C_n \times P_3) = \{0, 4, 8, \ldots, 3n\}$.*

Proof. Let g be a sign friendly labelling on the graph $C_n \times P_3$. Since $C_n \times P_3$ contains $3n = 3(4t) = 12t$ vertices and $5n = 5(4t) = 20t$ edges, we must have $e(+) = e(-) = 10t$. Denote the vertices as shown in the Figure 1.

Now label the edges $y_{4i-2}\, y_{4i-1}$ for $1 \leq i \leq t$, $y_{4i}\, y_{4i+1}$ for $1 \leq i \leq t-1$, $y_{4t}\, y_1$ by '$-$', $x_{4i-3}\, x_{4i-2}$ for $1 \leq i \leq t$, $x_{4i}\, x_{4i+1}$ for $1 \leq i \leq t-1$, $x_{4t}x_1$ by '$-$', $z_{4i-3}\, z_{4i-2}$ for $1 \leq i \leq t$, $z_{4i}\, z_{4i+1}$ for $1 \leq i \leq t-1$, $z_{4t}z_1$ by '$-$', $x_{2i-1}\, y_{2i-1}$ for $1 \leq i \leq 2t$, $y_{2i-1}\, z_{2i-1}$ for $1 \leq i \leq 2t$ by '$-$'. Label the remaining edges by '+'. Then we get $|v(-) - v(+)| = 12t$.

Now by interchanging the labels of edges $x_{4j-3}\, x_{4j-2}$ and $x_{4j-2}\, x_{4j-1}$, for $1 \leq j \leq t$ we get $|v(-) - v(+)| = 12t - 4j = 3n - 4j$. Again by interchanging the labels of edges $y_{4j-3}\,y_{4j-2}$ and $y_{4j-2}\,y_{4j-1}$ for $1 \leq j \leq t$ we found $|v(-) - v(+)| = 8t - 4j = 2n - 4j$. Finally interchanging the labels of $z_{4j-3}\,z_{4j-2}$ and $z_{4j-2}\,z_{4j-1}$ for $1 \leq j \leq t$ we get $|v(-) - v(+)| = 4t - 4j$. Thus $\{0, 4, 8, \ldots, 3n\}$ are elements of $SBIS(C_n \times P_3)$.

Since $v(-)$ is always even we have $v(+) = 12t - v(-)$ is also even. Therefore $\{2, 6, 10, \ldots, 3n-2\}$ are not elements of $SBIS(C_n \times P_3)$.

Theorem 2.2. *If $n \equiv 1(mod\ 4)$, then $SBIS(C_n \times P_3) = \{1, 3, 5, \ldots, 3n\}$.*
Proof. We assume for $t \geq 2$. Let g be an sign friendly labeling on the graph $C_n \times P_3$. Since $C_n \times P_3$ contains $3n = 3(4t+1) = 12t+3$ vertices and $5n = 5(4t+1) = 20t+5$ edges, we have two possibilities:
(i)$e(+) = 10t + 2$ and $e(-) = 10t + 3$ or
(ii) $e(+) = 10t + 3$ and $e(-) = 10t + 2$.
Case 1: Denote the vertices as shown in Figure 1. Now label the edges $x_{2i}\,x_{2i+1}$ for $1 \leq i \leq 2t$, $x_{3i-2}\,x_{3i-1}$ for $1 \leq i \leq t+1$, $z_{i+2}\,z_{i+3}$ for $1 \leq i \leq t+1$ and $z_{4t+1}\,z_1$ by '$-$'. Also $x_{i+1}\,y_{i+1}$ for $1 \leq i \leq 4t$, $y_1\,z_1$, $y_{3i}\,z_{3i}$ for $1 \leq i \leq t+1$ by '$-$'. Label the remaining edges by '+'. Then we obtain $v(-) = 0$ and $v(+) = 12t + 3$. Therefore $|v(-) - v(+)| = 12t + 3$.

Now by interchanging the labels of edges $x_{4j-3}\,x_{4j-2}$ and $x_{4j-2}x_{4j-1}$, for $1 \leq j \leq t$ we get $|v(-) - v(+)| = (12t + 3) - 4j$. Again by interchanging the labels of the edges $y_{3j-2}\,y_{3j-1}$ and $y_{3j-1}\,y_{3j}$ for $1 \leq j \leq t$ we get $|v(-) - v(+)| = (8t+3) - 4j$. Now by interchanging the labels of the edges $z_{3j-1}\,t_{3j}$ and $z_{3j}\,z_{3j+1}$ for $1 \leq j \leq t$ we get $|v(-) - v(+)| = (4t + 3) - 4j$. Therefore $\{3, 7, 10, \ldots, 3n\}$ are elements of $SBIS(C_n \times P_3)$.
Case 2: Label the edges $x_1\,x_2$, $x_{4t+1}\,x_1$, $z_1\,z_2$, $z_{4t+1}\,z_1$, $y_{4t+1}\,y_1$ by '$-$'. $x_{2i-1}\,y_{2i-1}$ for $1 \leq q \leq 2t$, $y_{2i+2}\,z_{2i+2}$ for $1 \leq i \leq 2t-1$ $y_{2i+1}\,y_{2i+2}$ for $1 \leq i \leq 2t-1$, $y_{2i-1}\,t_{2i-1}$ for $1 \leq i \leq 2t$, $x_{2i+2}\,y_{2i+2}$ for $1 \leq i \leq 2t-1$, by '$-$', label the remaining edges by '+'. Now we found that $v(+) = 1$ and $v(-) = 12t + 2$. Thus $|v(+) - v(-)| = 12t + 1 = 3n - 2$.

Now by interchanging the labels of the edges $x_{2j-1}\,x_{2j}$ and $x_{2j}\,y_{2j}$ for $1 \leq j \leq 2t$, we found $|v(+) - v(-)| = 12t + 1 - 4j = 4t + 1$. Again interchanging the labels of the edges $z_{2j}\,z_{2j+1}$ and $z_{2j+1}\,y_{2j+1}$ for $1 \leq j \leq t$ we get $|v(+) - v(-))| = 4t + 1 - 4j = 1$. Therefore $\{1, 5, 9, \ldots, 3n-2\}$ are elements of $SBIS(C_n \times P_3)$.

Theorem 2.3. *If $n \equiv 2(mod\ 4)$, then $SBIS(C_n \times P_3) = \{2, 6, 10, \ldots, 3n\}$.*
Proof. Let g be an sign friendly labelling on the graph $C_n \times P_3$. Since $C_n \times P_3$ contains $3n = 3(4t + 2) = 12t + 6$ vertices and $5n = 5(4t + 2) = 20t + 10$ edges, we must have $e(+) = e(-) = 10t + 5$.

Denote the vertices as shown in Figure 1. Now label the edges $x_{2i-1}\,x_{2i}$ for

$1 \leq i \leq 2t+1$, $y_i\,y_{i+1}$ for $1 \leq i \leq 4t+2$ and $y_i z_i$ for $1 \leq i \leq 4t+2$ by '$-$'. Label the remaining edges by '$+$'. Then we found that $v(-) = 12t+6$ and $v(+) = 0$. Therefore $|v(+) - v(-)| = 12t+6 = 3n$.

Now by interchanging the labels of the edges $x_{3j-2}\,x_{3j-1}$ and $x_{3j-1}\,x_{3j}$ for $1 \leq j \leq t+1$, we get $|v(+)-v(-)| = 12t+6-4j = 3n-4j$. Again by interchanging the labels of edges $y_{2j-1}\,z_{2j-1}$ and $z_{2j-1}z_{2j}$ for $1 \leq j \leq 2t$ we get $|v(+) - v(-)| = 8t + 2-4j$. Thus $\{2, 6, 10, \ldots, 3n\}$ are elements of $SBIS(C_n \times P_3)$. Since $v(-)$ is always even we have $v(+) = (12t+6) - v(-)$ is also even. Therefore $\{0, 4, 8, \ldots, 3n-2\}$ are not elements of $SBIS(C_n \times P_3)$.

Theorem 2.4. *If $n \equiv 3(mod\ 4)$, then $SBIS(C_n \times P_3) = \{1, 3, 5, \ldots, 3n\}$.*
Proof. Let g be an sign friendly labelling on the graph $C_n \times P_3$. Since $C_n \times P_3$ contains $3n = 3(4t+3) = 12t+9$ vertices and $5n = 5(4t+3) = 20t+15$ edges, we must have
(i)$e(+) = 10t+7$ and $e(-) = 10t+8$ or
(ii)$e(+) = 10t+8$ and $e(-) = 10t+7$.
Case 1: Denote the vertices as shown in Figure 1. Now label the edges $x_{2i}\,x_{2i+1}$ for $1 \leq i \leq 2t+1$, $y_i\,y_{i+1}$ for $1 \leq i \leq 4t+2$, $y_{4t+3}\,y_1$,$x_1\,y_1$ by '$-$',$y_i\,z_i$ for $2 \leq i \leq 4t+3$,$z_1 z_2$ by '$-$'. Label the remaining edges by '$+$'. Then we found that $|v(-) - v(+)| = 12t+7$.

Now by interchanging the labels of the edges $x_{3j-2}\,x_{3j-1}$ and $x_{3j-1}\,x_{3j}$ for $1 \leq j \leq 2t-1$, we get $|(12t+7) - 4j| = 4t+11 = n+8$. Again by interchanging the labels $y_{2j}z_{2j}$ and $z_{2j}\,z_{2j+1}$ for $1 \leq j \leq t+2$, we get $|v(-) - v(+)| = (4t+11) - 4j$. Thus $3, 7, 9, \ldots, 3n-2$ are elements of $SBIS(C_n \times P_3)$.
Case 2: Label the edges $x_{4i-1}\,x_{4i}$ for $1 \leq i \leq t$ $x_{4i}\,x_{4i+1}$ for $1 \leq i \leq t$, $x_{4t+3}\,x_1$, $y_{4i-1}\,y_{4i}$ for $1 \leq i \leq t$,$y_{4i}\,y_{4i+1}$ for $1 \leq i \leq t$ by '$-$'. $z_i z_{i+1}$ for $1 \leq i \leq 4t+2$, $z_{4t+3}\,z_1$ $y_{4t+3}\,y_1$,$y_i\,z_i$ for $1 \leq i \leq 4t+3$ by '$-$', label the remaining edges by '$+$'. Now we found that $v(-) = 0$ and $v(+) = 12t+9$. Thus $|v(-) - v(+)| = 12t+9 = 3n-2$

Now by interchanging the labels of the edges $y_{2j-1}\,z_{2j-1}$ and $z_{2j-1}z_{2j}$ for $1 \leq j \leq 2t+1$, we get $|v(+)-v(-)| = |(12t+9)-4j|$. Again by interchanging the labels of the edges $x_{4j}x_{4j+1}$ and $x_{4j+1}x_{4j+2}$ for $1 \leq j \leq t$ we get $|v(+)-v(-)| = (4t+5)-4j$. Interchanging the labels of the edges $x_{4t+3}x_1$ and x_1x_2 we get $|v(+) - v(-)| = 1$. Thus $1, 5, 9, \ldots, 3n$ are elements of $SBIS(C_n \times P_3)$.

3. Sign balance index set of a Double triangular snake graph

Definition 3.1. [4] *A double triangular snake graph, denoted by $D[T_n]$, is obtained from a path with vertices $y_1, y_2, \ldots, y_n$ by joining y_i and y_{i+1} to a new vertex x_i for $i = 1, 2, 3, \ldots, n-1$ and to a new vertex z_i for $i = 1, 2, 3, \ldots, n-1$.*

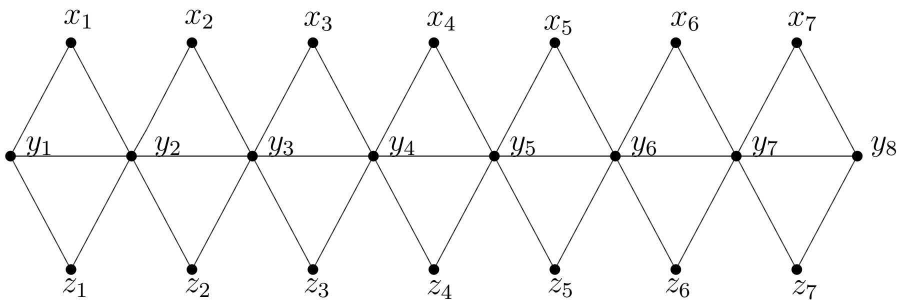

Figure 2: Double triangular snake graph $D[T_8]$

In this section we find the sign balance index set of a double triangular snake graph which consists of $3n-2$ vertices and $5(n-1)$ edges. We divide the problem of finding $SBIS(D[T_n])$ in to 8 cases, namely,

$n \equiv 0(mod\ 8)$, $n \equiv 1(mod\ 8)$, $n \equiv 2(mod\ 8)$, $n \equiv 3(mod\ 8)$,
$n \equiv 4(mod\ 8)$, $n \equiv 5(mod\ 8)$, $n \equiv 6(mod\ 8)$, $n \equiv 7(mod\ 8)$.

Theorem 3.2. *If* $n \equiv 0(mod\ 8)$ *then* $SBIS(D[T_n]) = \{2, 6, 10, \dots, 3n-2\}$.
Proof. We can easily check the result for $t = 0$, so we assume $t \geq 1$. Let f be a sign friendly labeling on $D[T_n]$. Since the graph contains $3n-2 = 24t-2$ vertices, $5(n-1) = 40t-5$ edges, we have two possibilities:
(i) $e(+) = 20t-3$, $e(-) = 20t-2$
(ii) $e(+) = 20t-2$, $e(-) = 20t-3$.

Now we consider the first case namely $e(+) = 20t-3$ and $e(-) = 20t-2$. Denote the vertices as shown in Figure 2. Now we label the edges $y_{2i-1}x_{2i-1}$, $y_{2i-1}\,z_{2i-1}$ for $1 \leq i \leq 4t$, $x_{2i}\,y_{2i+1}$, $z_{2i}\,y_{2i+1}$, $y_{2i}\,y_{2i+1}$ for $1 \leq i \leq 4t-1$ by '+' and label the remaining edges by '−'. Then we observe that $v(-) = 24t-2$ and there is no $+ve$ vertex in the graph. Thus $|\ v(-)-v(+)\ |= 24t-2 = 3n-2 = max\{SBIS(D[T_n])\}$.

By interchanging the labels of edges, $x_{2j-1}y_{2j-1}$ and $y_{2j-1}y_{2j}$, for $1 \leq j \leq 3t$ we get $|\ v(+)-v(-)\ |= 24t-4i-2$,
and by interchanging the labels of edges $z_{2j-1}y_{2j-1}$ and $z_{2j}y_{2j}$, for $1 \leq j \leq 3t-1$ we get $|\ v(+)-v(-)\ |= 12t-4i-2$.
Thus $2, 6, 10, \dots, 3n-2$ are elements of $SBIS(D[T_n])$.

Since $v(-)$ is always even we have $v(+) = (24t-2)-v(-)$ is also even. Therefore, the numbers $0, 4, 8, 12 \dots, 3n-4$ are not elements of $SBIS(D[T_n])$. Proof of the second case follows similarly.

Theorem 3.3. *If* $n \equiv 1(mod\ 8)$, *then* $SBIS(D[T_n]) = \{1, 3, 5, 7 \dots, 3n-2\}$.
Proof. Let g be a sign friendly labeling on $D[T_n]$. Since the graph contains $3n-2 = 24t+1$ vertices and $5(n-1) = 40t$ edges, we have $e(+) = e(-) = 20t$. We

label the edges $y_{2i}\, x_{2i}$, $y_{2i}\, z_{2i}$, $y_{2i+1}\, x_{2i}$, $y_{2i+1}\, z_{2i}$ for $1 \leq i \leq 3t$, $y_i\, y_{i+1}$ for $1 \leq i \leq 8t$ by '+' and label the remaining edges by '−'. Then we observe that $v(+) = 24t + 1$ and there is no $-ve$ vertex in the graph. Hence $\mid v(+) - v(-) \mid = 24t + 1 = max\{SBIS(D[T_n])\}$.

By interchanging the labels of the edges
$x_{2j-1}\, y_{2j-1}$ and $y_{2j-1} y_{2j}$, for $1 \leq j \leq 4t$ we get
$\mid v(+) - v(-) \mid = \mid 24t - 4j + 1 \mid$,
$z_{2j-1}\, y_{2j-1}$ and $y_{2j}\, z_{2j}$, for $1 \leq j \leq 3t$ we get
$\mid v(+) - v(-) \mid = \mid 8t - 4j + 1 \mid$,
$y_{2j+6t+1}\, y_{2j+6t-1}$ and $y_{2j}\, y_{2j+1}$, for $1 \leq j \leq t$ we get
$\mid v(+) - v(-) \mid = \mid -4t - 4j + 1 \mid$,
$y_{2j}\, y_{2j+1}$ and $y_{2j}\, x_{2j}$, for $1 \leq j \leq t$ we get
$\mid v(+) - v(-) \mid = \mid -8t - 4j + 1 \mid$,
$y_{2j+6t+1}\, z_{2j+6t}$ and $y_{2j+2t}\, y_{2j+2t+1}$, for $1 \leq j \leq t$ we get
$\mid v(+) - v(-) \mid = \mid -12t - 4j + 1 \mid$,
$y_{2j+2t}\, y_{2j+2t+1}$ and $y_{2j+2t}\, x_{2j+2t}$, for $1 \leq j \leq t$ we get
$\mid v(+) - v(-) \mid = \mid -16t - 4j + 1 \mid$
and finally interchanging the labels of edges
$y_{2j+6t+1} x_{2j+6t}$ and $y_{2j+4t} y_{2j+4t+1}$, for $1 \leq j \leq t$ we get
$\mid v(+) - v(-) \mid = \mid -20t - 4j + 1 \mid$.
Thus $SBIS(D[T_n]) = \{1, 3, 5, 7, \ldots, 3n - 2\}$.

Theorem 3.4. *If $n \equiv 2(mod\ 8)$, then $SBIS(D[T_n]) = \{0, 4, 8, 12, \ldots, 3n - 2\}$.*
Proof. Let g be a sign friendly labeling on $D[T_n]$. Since the graph contains $3n - 2 = 24t + 4$ vertices, $5(n - 1) = 40t + 5$ edges, we have two possibilities:
(i) $e(+) = 20t + 2$, $e(-) = 20t + 3$
(ii) $e(+) = 20t + 3$, $e(-) = 20t + 2$.

Now we consider the first case namely $e(+) = 20t+2$ and $e(-) = 20t+3$. Denote the vertices as shown in Figure 2. Now we label the edges $y_{2i-1}\, x_{2i-1}$, $y_{2i-1}\, z_{2i-1}$ for $1 \leq i \leq 4t + 1$, $x_{2i}\, y_{2i+1}$, $z_{2i}\, y_{2i+1}$, $y_{2i}\, y_{2i+1}$ for $1 \leq i \leq 4t$ by '+' and label the remaining edges by '−'. Then we observe that $v(-) = 24t + 4$ and there is no $+ve$ vertex in the graph. Thus $\mid v(-) - v(+) \mid = 24t+4 = 3n-2 = max\{SBIS(D[T_n])\}$.

By interchanging the labels of the edges
$x_{2j-1}\, y_{2j-1}$ and $y_{2j-1}\, y_{2j}$, for $1 \leq j \leq 3t + 1$ we get $\mid v(+) - v(-) \mid = 24t - 4j + 4$,
again by interchanging the labels of edges
$z_{2j-1}\, y_{2j-1}$ and $z_{2j}\, y_{2j}$, for $1 \leq j \leq 3t$ we get $\mid v(+) - v(-) \mid = 12t - 4j + 4$.
Thus $0, 4, 8, \ldots, 3n - 2$ are elements of $SBIS(D[T_n])$.

Since $v(-)$ is always even we have $v(+) = (24t + 4) - v(-)$ is also even. Therefore, the numbers $2, 6, 10, \ldots, 3n - 4$ are not elements of $SBIS(D[T_n])$. Proof of the second case follows similarly.

Theorem 3.5. *If* $n \equiv 3(mod\ 8)$*, then* $SBIS(D[T_n]) = \{1, 3, 5, 7, \ldots, 3n-2\}$.
Proof. Let g be a sign friendly labeling on $D[T_n]$. Since the graph contains $3n-2 = 24t+7$ vertices, $5(n-1) = 40t+10$ edges, we have $e(+) = e(-) = 20t+5$. Denote the vertices as shown in Figure 2. Label the edges $y_{2i}\,x_{2i}$, $y_{2i}\,z_{2i}$, $y_{2i+1}\,x_{2i}$, $y_{2i+1}\,z_{2i}$ for $1 \leq i \leq 3t$, $y_i\,y_{i+1}$ for $1 \leq i \leq 8t-1$, $y_{i+8t}\,z_{i+8t}$, $y_{i+8t+1}\,z_{i+8t}$ for $0 \leq i \leq 2$ by '+' and label the remaining edges by '−'. Then we observe that $v(+) = 24t+7$ and there is no $-ve$ vertex in the graph. Hence $\mid v(+) - v(-) \mid = 24t+7 = max\{SBIS(D[T_n])\}$.

By interchanging the labels of the edges
$x_{2j-1}\,y_{2j-1}$ and $y_{2j-1}\,y_{2j}$, for $1 \leq j \leq 4t$ we get
$\mid v(+) - v(-) \mid = \mid 24t - 4j + 7 \mid$,
$z_{2j-1}y_{2j-1}$ and $y_{2j}z_{2j}$, for $1 \leq j \leq 3t$ we get
$\mid v(+) - v(-) \mid = \mid 8t - 4j + 7 \mid$,
$x_{2j+6t}\,y_{2j+6t+1}$ and $y_{2j}y_{2j+1}$, for $1 \leq j \leq t+1$ we get
$\mid v(+) - v(-) \mid = \mid -4t - 4j + 7 \mid$,
$y_{2j}y_{2j+1}$ and $y_{2j}\,x_{2j}$, for $1 \leq j \leq t+1$ we get
$\mid v(+) - v(-) \mid = \mid -8t - 4j + 3 \mid$,
$y_{2j+6t-1}\,z_{2j+6t-1}$ and $y_{2j+3t}\,y_{2j+3t+1}$, for $1 \leq j \leq t$ we get
$\mid v(+) - v(-) \mid = \mid -12t - 4j - 1 \mid$,
$y_{2j+3t}\,y_{2j+3t+1}$ and $y_{2j+3t}\,x_{2j+3t}$, for $1 \leq j \leq t$ we get
$\mid v(+) - v(-) \mid = \mid -16t - 4j - 1 \mid$,
$y_{2j+6t+1}\,z_{2j+6t}$ and $y_{2j+5t}\,x_{2j+5t}$, for $1 \leq j \leq t-1$ we get
$\mid v(+) - v(-) \mid = \mid -20t - 4j - 1 \mid$,
$x_{8t+1}y_{8t+1}$ and $z_{8t+1}y_{8t+2}$ we get $\mid v(+) - v(-) \mid = \mid -24t - 1 \mid$
and finally interchanging the labels of edges
$y_{8t}\,y_{8t+1}$ and $y_{8t}z_{8t}$ we get $\mid v(+) - v(-) \mid = \mid -24t - 5 \mid$.
Thus $SBIS(D[T_n]) = \{1, 3, 5, 7 \ldots, 3n-2\}$.

Theorem 3.6. *If* $n \equiv 4(mod\ 8)$*, then* $SBIS(D[T_n]) = \{2, 6, 10, \ldots, 3n-2\}$.

Theorem 3.7. *If* $n \equiv 5(mod\ 8)$*, then* $SBIS(D[T_n]) = \{1, 3, 5, 7, \ldots, 3n-2\}$.

Theorem 3.8. *If* $n \equiv 6(mod\ 8)$*, then* $SBIS(D[T_n]) = \{0, 4, 8, 12, \ldots, 3n-2\}$.

Theorem 3.9. *If* $n \equiv 7(mod\ 8)$*, then* $SBIS(D[T_n]) = \{1, 3, 5, 7, \ldots, 3n-2\}$.

As the proofs of Theorems 3.6, 3.7, 3.8 and 3.9 are same as that of Theorems 3.2, 3.3, 3.4 and 3.5 we omit the proofs.

4. The sign balance index set of $\mu(P_n)$

Definition 4.1. *For a graph $G = (V, E)$, the Mycielskian of G is the graph $\mu(G)$ with vertex set consisting of the disjoint union $V \cup V' \cup \{u\}$, where $V' = \{x' : x \in V\}$ and edge set $E \cup \{x'y : xy \in E\} \cup \{x'u : x' \in V'\}$.*

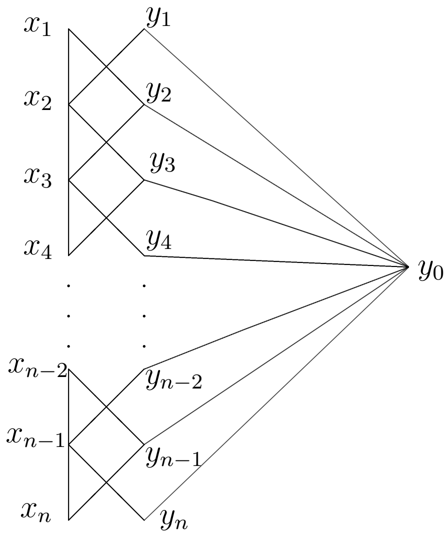

Figure 3: Mycielskian graph of a path P_n

In this section we consider the Mycielskian graph of path P_n $(n \geq 2)$, which consists of $2n+1$ vertices and $4n-3$ edges. Now we divide the problem of finding $SBIS(\mu(P_n))$ into two cases, namely,

$$n \equiv 0(mod\ 2) \qquad \text{and} \qquad n \equiv 1(mod\ 2),$$

Theorem 4.2. *If $n \equiv 0(mod\ 2)$, then the $SBIS(\mu(P_n)) = \{1, 3, 5, \ldots, 2n+1\}$.*
Proof. Let g be an sign friendly labeling on $\mu(P_n)$. Since the graph contains $2n+1 = 4t+1$ vertices, $4n-3 = 8t-3$ edges, we have two possibilities:
(i) $e(+) = 4t-1$, $e(-) = 4t-2$
(ii) $e(+) = 4t-2$, $e(-) = 4t-1$.

Now we consider the first case i.e., $e(+) = 4t-1$ and $e(-) = 4t-2$. Denote the vertices of $\mu(P_n)$ as in the Figure 3. Now we label the edges $x_{2i-1}\, y_{2i}$, $x_{2i+1}\, y_{2i}$ for $1 \leq i \leq t-1$, $x_i\, x_{i+1}$ for $1 \leq i \leq 2t-3$, $x_{2t-2}\, y_{2t-1}$, $x_{2t}\, y_{2t-1}$ and $x_{2t-1}\, x_{2t}$ by '$-$' and label the remaining edges by '$+$'. Then we observe that $v(+) = 4t+1$ and there is no $-ve$ vertex in the graph. Thus $\mid v(-) - v(+) \mid = 4t+1 = 2n+1 = max\{SBIS(\mu(P_n))\}$.
By interchanging the labels of edges

$x_{2j}\, x_{2j+1}$ and $x_{2j}\, y_{2j+1}$ for $1 \leq j \leq t-2$ we get $\mid v(+) - v(-) \mid = \mid 4t - 4j + 1 \mid$,
$x_{2t-1}x_{2t}$ and $x_{2t-1}y_{2t}$ we get $\mid v(+) - v(-) \mid = \mid -4 + 9 \mid = 5$,
x_1y_2 and y_1y_0, x_2y_3 and y_2y_0 we get $\mid v(+) - v(-) \mid = 1$,
x_3y_2 and y_3y_0, x_3y_4 and y_4y_0 we get $\mid v(+) - v(-) \mid = 3$,
x_4y_5 and y_5y_0, x_5y_4 and y_6y_0 we get $\mid v(+) - v(-) \mid = 7$,
x_5y_6 and y_7y_0, x_6y_7 and y_8y_0 we get $\mid v(+) - v(-) \mid = 11$,
$x_{2\lfloor \frac{j-1}{2} \rfloor + 7} y_{2\lceil \frac{j-1}{2} \rceil + 6}$ and $y_{2j+7}y_0$, $x_{2j+6}y_{2j+7}$ and $y_{2j+8}y_0$ for $1 \leq j \leq t-5$
we get $\mid v(+) - v(-) \mid = \mid -4j - 11 \mid$,
$x_{2\lfloor \frac{t-5}{2} \rfloor + 7} y_{2\lceil \frac{t-5}{2} \rceil + 6}$ and $x_{2t-2}x_{2t-1}$ we get $\mid v(+) - v(-) \mid = \mid -4t - 4 + 9 \mid = 4t - 5$
Finally by interchanging the labels of the edges $x_{2\lfloor \frac{t-4}{2} \rfloor + 7} y_{2\lceil \frac{t-4}{2} \rceil + 6}$ and $x_{2t-1}y_0$ we get $\mid v(+) - v(-) \mid = \mid -4t - 4 + 5 \mid = 4t - 1$,
Thus $SBIS(\mu(P_n)) = \{1, 3, 5, \ldots, 2n + 1\}$. Proof of the second case follows similarly.

Theorem 4.3. *If $n \equiv 1 (mod\ 2)$, then the $SBIS(\mu(P_n)) = \{1, 3, 5, \ldots, 2n + 1\}$.*
Proof. Let g be an sign friendly labeling on $\mu(P_n)$. Since the graph contains $2n + 1 = 4t + 3$ vertices, $4n - 3 = 8t + 1$ edges, we have two possibilities: (i) $e(+) = 4t + 1$, $e(-) = 4t$ (ii) $e(+) = 4t$, $e(-) = 4t + 1$. Now we consider the first case i.e., $e(+) = 4t + 1$ and $e(-) = 4t$. Denote the vertices of $\mu(P_n)$ as shown in Figure 3. Now we label the edges $x_{2i-1}y_{2i}$, $x_{2i+1}y_{2i}$ for $1 \leq i \leq t$ and x_ix_{i+1} for $1 \leq i \leq 2t$ by '$-$' and label the remaining edges by '$+$'. Then we observe that $v(+) = 4t + 3$ and there is no $-ve$ vertex in the graph. Thus $\mid v(-) - v(+) \mid = 4t + 3 = 2n + 1 = max\{SBIS(\mu(P_n))\}$.
By interchanging the labels of edges
$x_{2j}x_{2j+1}$ and $x_{2j}y_{2j+1}$ for $1 \leq j \leq t$ we get $\mid v(+) - v(-) \mid = \mid 4t - 4j + 3 \mid$,
$x_{2t}y_{2t+1}$ and $y_{2t+1}y_0$ we get $\mid v(+) - v(-) \mid = 1$,
x_1y_2 and y_1y_0, x_2y_3 and y_2y_0 we get $\mid v(+) - v(-) \mid = 5$,
x_3y_2 and y_3y_0, x_3y_4 and y_4y_0 we get $\mid v(+) - v(-) \mid = 9$,
x_4y_5 and y_5y_0, x_5y_4 and y_6y_0 we get $\mid v(+) - v(-) \mid = 13$,
x_5y_6 and y_7y_0, x_6y_7 and y_8y_0 we get $\mid v(+) - v(-) \mid = 17$
and finally by interchanging the labels of edges
$x_{2\lfloor \frac{j-1}{2} \rfloor + 7} y_{2\lceil \frac{j-1}{2} \rceil + 6}$ and $y_{2j+7}y_0$, $x_{2j+6}y_{2j+7}$ and $y_{2j+8}y_0$ for $1 \leq j \leq t-4$
we get $\mid v(+) - v(-) \mid = \mid -4j - 17 \mid$.
Thus $SBIS(\mu(P_n)) = \{1, 3, 5, \ldots, 2n + 1\}$. Proof of the second case follows similarly.

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